Published by Institute of Physics Publishing for SISSA/ISAS
Received: *April 18, 2002*
Accepted: *May 17, 2002*

Exactly marginal deformations of $\mathcal{N}=4$ SYM and of its supersymmetric orbifold descendants

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Abstract: In this paper we study *exactly* marginal deformations of field theories living on D3-branes at low energies. These theories include $\mathcal{N}=4$ supersymmetric Yang-Mills theory and theories obtained from it via the orbifolding procedure. We restrict ourselves only to orbifolds and deformations which leave some supersymmetry unbroken. A number of new families of $\mathcal{N}=1$ superconformal field theories are found. We analyze the deformations perturbatively, and also by using general arguments for the dimension of the space of *exactly* marginal deformations. We find some cases where the space of perturbative *exactly* marginal deformations is smaller than the prediction of the general analysis (at least up to three-loop order), and other cases where the perturbative result (at low orders) has a non-generic form.

Keywords: Renormalization Group, D-branes, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

In recent years there has been a great burst of research and interest in $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. Much of the interest is due to the fact that this theory has a string theory dual via the AdS/CFT correspondence (see [1, 2]). $\mathcal{N} = 4$ SU(N) SYM appears in this context as a low-energy effective description of N coincident D3-branes. By looking at D3-branes at an orbifold point one can obtain [3, 4, 5] effective descriptions in terms of conformal field theories with less supersymmetries ($\mathcal{N} = 2$, $\mathcal{N} = 1$, and perhaps also $\mathcal{N} = 0$). It is interesting to look at *exactly* marginal deformations of such theories. In the AdS/CFT [1] correspondence such deformations on the field theory side correspond to moduli of the string theory. For instance [5], in the $\mathcal{N} = 2$ $\mathbb{Z}_k$ orbifold theory there are k *exactly* marginal deformations which preserve the $\mathcal{N} = 2$ SUSY. These deformations correspond to the string coupling and ALE blow-up modes on the string theory side.

In this work we investigate marginal deformations, which preserve at least $\mathcal{N} = 1$ supersymmetry, of $\mathcal{N} = 4$ SYM and its orbifold $\mathcal{N} = 2$ and $\mathcal{N} = 1$ descendants. There are two known *exactly* marginal deformations of this type for $\mathcal{N} = 4$ SYM (see [6] and references therein),[1] and we show that these (and the gauge coupling) are the only *exactly* marginal supersymmetric deformations of this theory. The planar diagram contribution in the orbifold theories is the same as in the $\mathcal{N} = 4$ theory [9, 10], so in the large N limit many correlation functions in these theories coincide up to some gauge coupling rescaling. Thus, one could expect that the orbifold theories possess similar *exactly* marginal deformations. We will show that this is actually the case even without going to the large N limit, and we find additional *exactly* marginal operators from the twisted sectors.

[1]Marginal deformations of $\mathcal{N} = 4$ SYM were also discussed in [7, 8]

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In some cases we find that the dimension of the space of *exactly* marginal deformations at low orders in perturbation theory is smaller than the general analysis implies. For SU($N=3$) gauge groups more deformations are possible, and we find a much larger number of *exactly* marginal deformations.

One motivation for studying such families of conformal theories is that they can then be used as a starting point for renormalization group (RG) flows by turning on additional relevant operators. For instance, one might hope that by starting from a theory obtained by an *exactly* marginal deformation of $\mathcal{N}=2$ orbifold theories, one could flow to duality cascades of the type considered in [11],[2] for which a direct field theory definition is not known. Unfortunately, we do not find any deformations which are useful for this.

The reason that we constrain ourselves to working with supersymmetric field theories is the existence of relations between β-functions in these theories, due to non-renormalization theorems. The β-functions in supersymmetric field theories can be expressed in terms of the anomalous dimensions, γ. For the superpotential couplings this is a consequence of the superpotential non-renormalization theorem, and for the gauge coupling it is the NSVZ β-function [12, 13, 14]. For a superpotential $W=\frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k$ the beta function is given by

$$\beta_{Y^{ijk}} = Y^{p(ij}\gamma_p^{k)} = Y^{ijp}\gamma_p^k + (k\leftrightarrow i) + (k\leftrightarrow j)\,, \tag{1.1}$$

and the NSVZ β-function for the gauge coupling is:[3]

$$\beta_g = \frac{g^3}{16\pi^2}\left[\frac{Q-2r^{-1}\mathrm{Tr}\left[\gamma C(R)\right]}{1-2C_1 g^2(16\pi^2)^{-1}}\right]. \tag{1.2}$$

The strategy of our search for *exactly* marginal deformations is the following. We first make a generic computation of the expected dimension of the space of *exactly* marginal deformations by using the relations between the β-functions and the anomalous dimensions above, as in [6] and references therein. Then we check in perturbation theory whether these deformations actually appear, and whether their form agrees with the general analysis.

In the next section we study *exactly* marginal deformations of $\mathcal{N}=4$ SYM theory. In section 3 we describe the orbifold procedure and study the *exactly* marginal deformations of the orbifold theories. The $\mathcal{N}=2$ case is studied in detail in section 3.1.2. Details of other cases may be found in [15].

2. $\mathcal{N}=4$ Super-Yang-Mills theory

Four dimensional $\mathcal{N}=4$ SU(N) SYM theory appears as a low energy description of the physics on N coincident D3-branes in type-IIB superstring theory. In $\mathcal{N}=1$ superspace notation the matter content is three chiral superfields, Φ^i, in the adjoint representation of the gauge group. Besides changing the gauge coupling, the only classically marginal

[2] O. A. would like to thank I. Klebanov and J. Polchinski for discussions on this issue.

[3] Here Q is the one loop gauge β-function, $C(R)^i_j = (R_A R_A)^i_j$ where R_A is the representation of the matter chiral superfields, $r=\delta_{AA}$, and $C_1\delta_{AB} = f_{ACD}f_{BCD}$.

deformations (for $N > 2$) preserving $\mathcal{N} = 1$ SUSY are superpotentials of the form

$$\begin{aligned}&\frac{i\lambda\sqrt{2}}{3!}\epsilon_{ijk}\mathrm{Tr}\left(\Phi^i\left[\Phi^j,\Phi^k\right]\right),\\ &\frac{h_{ijk}}{3!}\mathrm{Tr}\left(\Phi^i\left\{\Phi^j,\Phi^k\right\}\right).\end{aligned} \tag{2.1}$$

Traces are taken in the fundamental representation of the gauge group. In the $\mathcal{N} = 4$ theory the λ coupling is equal to the gauge coupling and all h_{ijk} vanish.

From (1.1), (1.2) we obtain:

$$\begin{aligned}\beta_g, \beta_\lambda &\propto \mathrm{Tr}(\gamma)\,,\\ \beta_{h_{ijk}} &\propto h_{p(ij}\gamma^p_{k)}\,.\end{aligned} \tag{2.2}$$

If we turn on only λ, $h_{111} = h_{222} = h_{333}$ and h_{123}, then $\beta_{h_{ijk}} \propto Tr\gamma$ as well, and thus the single equation $\mathrm{Tr}(\gamma) = 0$ is enough to ensure conformal invariance. We get one equation for four coupling constants, so we expect a three dimensional manifold of fixed points [6]. We will see that these are the only *exactly* marginal supersymmetric deformations in this theory (up to the global SU(3) symmetry we have between the three Φ^i).

The one-loop calculation of the beta functions and the anomalous dimensions gives:[4]

$$\begin{aligned}\gamma^{(1)ai}_{bj} &= \frac{1}{16\pi^2}\left\{2C_1\left(\lambda^2 - g^2\right)\delta_{ij} + \frac{N^2-4}{N}C_2^3 h^{(2)}_{ij}\right\}\delta_{ab},\\ \beta^{(1)}_\lambda &= \frac{\lambda}{16\pi^2}\left\{6C_1\left(\lambda^2 - g^2\right) + \frac{N^2-4}{N}C_2^3\mathrm{Tr}\left(h^{(2)}\right)\right\},\\ \beta^{(1)}_{h_{ijk}} &= \frac{1}{16\pi^2}\left\{6C_1\left(\lambda^2 - g^2\right)h_{ijk} + \frac{N^2-4}{N}C_2^3 h^{(3)}_{ijk}\right\},\end{aligned} \tag{2.3}$$

where we defined:

$$\begin{aligned}h^{(3)}_{ijk} &\equiv h^*_{plm}\left(h_{ijp}h_{klm} + h_{kjp}h_{ilm} + h_{ikp}h_{jlm}\right)\\ h^{(2)}_{ij} &\equiv h_{ilm}h^*_{jlm}\,.\end{aligned} \tag{2.4}$$

The equations simplify if we rescale the coupling constants:

$$g \to \frac{\sqrt{C_1}}{4\pi}g\,, \qquad \lambda \to \frac{\sqrt{C_1}}{4\pi}\lambda \qquad \text{and} \qquad h_{ijk} \to \frac{\sqrt{C_2^3}}{4\pi}\sqrt{\frac{N^2-4}{N}}h_{ijk}\,. \tag{2.5}$$

The β-functions become:

$$\beta_g = -\frac{2g^3}{1-2g^2}\mathrm{Tr}(\gamma)\,, \qquad \beta_\lambda = \lambda\mathrm{Tr}(\gamma)\,. \tag{2.6}$$

Here the trace is taken only over the SU(3) indices and not over gauge indices.

[4] Here $C_2\delta_{ab} \equiv \mathrm{Tr}(T_aT_b)$ where T_a are SU(N) generators in the fundamental representation, and a, b is an adjoint representation index.

From these β-functions we can obtain a differential equation:

$$-\frac{1}{2g^3}dg + \frac{1}{g}dg = \frac{d\lambda}{\lambda}\,. \tag{2.7}$$

This can be easily solved to give:

$$\frac{\lambda}{\lambda_0} = \frac{g e^{1/4g^2}}{g_0 e^{1/4g_0^2}}\,. \tag{2.8}$$

This result means that the RG flow lines in the $\lambda - g$ plane are exactly known (to the extent that we can count on the NSVZ β-function). It is easy to convince oneself that there is no line with both couplings going to zero in the UV, except the trivial case when one of the couplings is constantly zero. This implies that there is no choice of coupling constants for which this theory is asymptotically free.

In order to have a fixed point we have to satisfy $\mathrm{Tr}(\gamma) = 0$, which implies at one loop that:

$$\mathrm{Tr}(h^{(2)}) = -6\left(\lambda^2 - g^2\right)\,, \tag{2.9}$$

and we can substitute this into $\beta_{h_{ijk}}$ to get another condition:

$$\mathrm{Tr}\left(h^{(2)}\right) h_{ijk} = h^{(3)}_{ijk}\,. \tag{2.10}$$

By multiplying (2.10) on both sides by h^*_{ijk} we get:

$$3\mathrm{Tr}\left(\left(h^{(2)}\right)^2\right) = \left(\mathrm{Tr}\left(h^{(2)}\right)\right)^2, \tag{2.11}$$

which implies $h^{(2)}_{ij} = \alpha^2\delta_{ij}$, and then γ is proportional to the identity matrix. One can show that this implies that we turn on only the λ, $h_{111} = h_{222} = h_{333}$ and h_{123} couplings (or their SU(3) rotations), and that $\alpha^2 = \frac{1}{3}\sum_{i,j,k}|h_{ijk}|^2$.

The fixed points we found are IR stable fixed points, since we have:

$$\mathrm{Tr}(\gamma) = 3\left(2\left(\lambda^2 - g^2\right) + \alpha^2\right), \tag{2.12}$$

and the condition for a fixed point is $\mathrm{Tr}(\gamma) = 0$. From the β-functions we calculated we see that if we increase one of the couplings λ or h_{ijk}, $\mathrm{Tr}(\gamma)$ becomes positive thus decreasing these couplings and increasing the gauge coupling in IR, till we get again zero. A similar behavior arises if we decrease the couplings. Thus, we conclude that in the weak coupling limit all fixed points that exist imply diagonal γ and are IR stable.

The general analysis above implies that the three dimensional surface of fixed points should persist also at strong coupling. The gravitational dual of these *exactly* marginal deformations will be discussed in [16] (it was also discussed in [17]).

3. Orbifold theories

It is possible to reduce the number of supersymmetries of the $d = 4$ $\mathcal{N} = 4$ SYM, which we discussed in the previous section, via the orbifolding procedure, by looking at the theories arising from D3-branes at orbifold singularities [3, 4, 5].

We will look at N coincident D3-branes at the $\mathbb{Z}_k$ orbifold singularity of a $\mathbb{C}^3/\mathbb{Z}_k$ space. Denoting the $\mathbb{C}^3$ coordinates transverse to the D3-branes by Z^l, the orbifold group acts on them as

$$Z^l \to \omega^{a_l} Z^l \,, \tag{3.1}$$

where $\omega \equiv e^{\frac{2\pi i}{k}}$ and (a_1, a_2, a_3) is a triple of integers.[5]

To see how the orbifold acts on the D-branes we put kN D3-branes on the covering space and group them in N sets of k branes. We put each set of D-branes in the regular representation of $\mathbb{Z}_k$, which is the direct sum of k one dimensional irreducible representations parameterized by the integer n, given by ω^n (with ω defined above). We denote each brane by a pair of indices, $i = 0, \ldots, N-1$ and $I = 0, \ldots, k-1$.

The bosonic fields on the D3-branes are a gauge field and three complex scalars Q_i, sitting in chiral multiplets, whose eigenvalues label the Z^i positions of the branes. The projection on the gauge fields is given by (we write the adjoint fields in double index notation, with the upper index in the fundamental representation and the lower in the anti-fundamental):

$$A^{I,i}_{J,j} = \omega^{I-J} A^{I,i}_{J,j} \,, \tag{3.2}$$

and the projection on the chiral multiplets (which are related to the 6 transverse directions) is:

$$(Q_l)^{I,i}_{J,j} = \omega^{I-J+a_l} (Q_l)^{I,i}_{J,j} \,. \tag{3.3}$$

This projection comes from acting on the Chan-Paton indices as well as on the space-time index.

The gauge fields which survive the projection are $A^{I,i}_{J,j}$ where both indices lie in the same irreducible representation of $\mathbb{Z}_k$, $I = J$, giving a total of k copies of $\mathrm{U}(N)$.[6] The matter content of the theory consists of chiral superfields in the following representations:

$$\begin{aligned}
&\oplus_{I=0}^{k-1} \left(1, 1, \ldots, 1, N_{(I)}, 1, \ldots, 1, \bar{N}_{(I+a_1)}, 1, \ldots, 1, 1\right) , \\
&\oplus_{I=0}^{k-1} \left(1, 1, \ldots, 1, N_{(I)}, 1, \ldots, 1, \bar{N}_{(I+a_2)}, 1, \ldots, 1, 1\right) , \\
&\oplus_{I=0}^{k-1} \left(1, 1, \ldots, 1, N_{(I)}, 1, \ldots, 1, \bar{N}_{(I+a_3)}, 1, \ldots, 1, 1\right) ,
\end{aligned} \tag{3.4}$$

[5] The vector $\vec{a}$ has to satisfy $\sum_i a_i = 0 \pmod{k}$ in order that the orbifold action will be part of SU(3) and not U(3), which is the condition for preserving supersymmetry. If we choose all the components to be non zero we have $\mathbb{Z}_k \subset \mathrm{SU}(3)$ thus leaving us with one supersymmetry. If we choose one of the components zero then we can have $\vec{a} = (n, 0, -n) \pmod{k}$. This case is equivalent to the $(1, 0, -1)$ case, and in this case $\mathbb{Z}_k \subset \mathrm{SU}(2)$ and we have $\mathcal{N} = 2$ supersymmetry. So there is only one choice giving an $\mathcal{N} = 2$ theory here. If two components of $\vec{a}$ vanish the remaining one must also vanish, giving the $\mathcal{N} = 4$ case.

[6] We will treat the gauge groups as $\mathrm{SU}(N)$ rather than $\mathrm{U}(N)$ from here on, since one U(1) factor is decoupled and the others are free in the IR.

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where $N_{(I)}$ labels the I'th $\mathrm{SU}(N)$ factor, except when $a_I = 0$ in which case we get a chiral superfield in the adjoint of $\mathrm{U}(N)^k$. We will denote these fields by Q_l^I where $I \in (0, \ldots, k-1)$, $l \in (1, 2, 3)$. The index I denotes the index of the $\mathrm{SU}(N)$ group of which the field is in the fundamental (or adjoint) representation.

The matter content of this theory can be summarized in a "quiver" diagram, where vertices represent the gauge groups, and oriented lines represent chiral multiplets in the fundamental of the group to which they point and the antifundamental of the second group.

In this paper we won't be interested in particular in the superpotential coming from the orbifold theory, but rather in the most general superpotential with this matter content which is classically marginal. Generally, the only possible marginal superpotential[7] is of the form:

$$W = h^I_{lmn} \mathrm{Tr}\left(Q_l^I Q_m^{I+a_l} Q_n^{I+a_l+a_m}\right), \tag{3.5}$$

with $a_l + a_m + a_n = 0 \pmod{k}$. The definition of the couplings h^I_{lmn} in this way is redundant, since

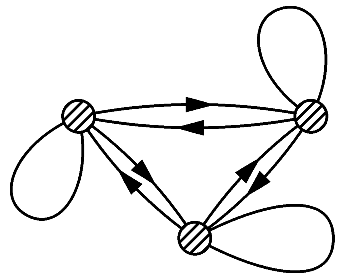

Figure 1: $\mathbb{C}^3/\mathbb{Z}_3$ (1,-1,0) quiver diagram

$$h^I_{lmn} = h^{I+a_l}_{mnl} = h^{I+a_l+a_m}_{nlm}\,. \tag{3.6}$$

Obviously, for a general choice of k and of $\vec{a}$, the only possibility is to take (l, m, n) to be some permutation of $(1, 2, 3)$. In some special cases additional superpotentials are possible. For $N > 3$ these cases are :

- $\mathcal{N} = 1$ SUSY
  - general k, $\vec{a} = (a, a, -2a)$,
  - $k = 3k'$, $\vec{a} = (a, k/3 - a, -k/3)$,
  - $k = 3k'$, $\vec{a} = (k/3, k/3, k/3)$,
  - $k = 6k'$, $\vec{a} = (k/6, k/6, 2k/3)$.
- $\mathcal{N} = 2$ SUSY
  - $k = 3$, $\vec{a} = (1, -1, 0)$.

When a_1, a_2 and k have a common divisor J larger than one, then the theory splits into J copies of the $\mathbb{Z}_{k/J}$, $\frac{1}{J}\vec{a}$ theory. In particular there is no meaning to discussing the $\vec{a} = (k/3, k/3, k/3)$, $\vec{a} = (k/6, k/6, 2k/3)$ theories for general k, they are all equivalent to the ones with $k = 3, 6$ respectively. We will assume that a_1, a_2 and k have no non-trivial common divisor.

[7] In the quiver diagram, the possible superpotentials (for $\mathrm{SU}(N > 3)$) correspond to oriented triangles.

When the gauge group is $\mathrm{SU}(3)^k$ there is another possible set of superpotentials. If Q_i^I, Q_j^I, Q_p^I are bifundamentals of the same two groups, we can add the following marginal operator:

$$W = \frac{\rho_{ijp}^I}{3!}\epsilon_{lmn}\epsilon^{abc}\left(Q_i^I\right)_a^l \left(Q_j^I\right)_b^m \left(Q_p^I\right)_c^n . \tag{3.7}$$

We will start by treating the most general case in some detail, and then discuss briefly the special cases. Only fields in the same representation can mix under renormalization, so we can write the gamma matrix as γ_{lm}^I, where γ_{lm}^I can be non vanishing only if $a_l = a_m$. In all the orbifold theories the 1-loop beta functions vanish, and the NSVZ formula gives

$$\beta_{g_I} \propto \mathrm{Tr}\left(\gamma^I\right) + \gamma_{11}^{I-a_1} + \gamma_{22}^{I-a_2} + \gamma_{33}^{I-a_3} . \tag{3.8}$$

For the superpotential couplings we have the usual expression coming from the general formula (1.1),

$$\beta_{h_{lmn}^I} \propto h_{pmn}^I \gamma_{pl}^I + h_{lpn}^I \gamma_{pm}^{I+a_l} + h_{lmp}^I \gamma_{pn}^{I+a_l+a_m} . \tag{3.9}$$

3.1 The general case

As we mentioned at the beginning of this section, in the most general case h_{ijk}^I (from (3.5)) can appear only if (i,j,k) is some permutation of $(1,2,3)$. The redundancy condition gives

$$\begin{aligned} h_{123}^I &= h_{231}^{I+a_1} = h_{312}^{I+a_1+a_2} , \\ h_{132}^I &= h_{321}^{I+a_1} = h_{213}^{I+a_1+a_3} , \end{aligned} \tag{3.10}$$

so we actually have only $2 \times k$ independent superpotential couplings here, $h_I \equiv h_{123}^I$ and $h_I' \equiv h_{132}^I$.

We start with a general analysis of the expected dimension of the space of *exactly* marginal deformations, along the lines of [6]. From the general β-functions (3.8), (3.9), we obtain, using the fact that in this case there is no mixing between the fields,

$$\begin{aligned} \beta_{h_I} &\propto \gamma_1^I + \gamma_2^{I+a_1} + \gamma_3^{I+a_1+a_2} , \\ \beta_{h_I'} &\propto \gamma_1^I + \gamma_3^{I+a_1} + \gamma_2^{I+a_1+a_3} , \\ \beta_{g_I} &\propto \gamma_1^I + \gamma_2^I + \gamma_3^I + \gamma_1^{I+a_2+a_3} + \gamma_2^{I+a_1+a_3} + \gamma_3^{I+a_1+a_2} . \end{aligned} \tag{3.11}$$

Naively the vanishing of (3.11) gives $3k$ conditions for $3k$ couplings so we don't expect any *exactly* marginal directions. However, there are some relations between the β-functions.

Let us denote the largest common divisor of k and a_i by α_i, and define $\mathcal{S}_{a_i}^J$ to be the set of indices $(J, J+a_i, J+2a_i, \ldots)$. We find that

$$\begin{aligned} \sum_{I\in\mathcal{S}_{a_1}^J} \frac{\beta_{h_I}}{h_I} &\propto \sum_{I\in\mathcal{S}_{a_1}^J} \left(\gamma_1^I + \gamma_2^I\right) + \sum_{I\in\mathcal{S}_{a_1}^{J+a_2}} \gamma_3^{I+a_2} , \\ \sum_{I\in\mathcal{S}_{a_1}^J} \frac{\beta_{h_I'}}{h_I'} &\propto \sum_{I\in\mathcal{S}_{a_1}^J} \left(\gamma_1^I + \gamma_3^I\right) + \sum_{I\in\mathcal{S}_{a_1}^{J+a_1+a_3}} \gamma_2^{I+a_1+a_3} , \\ \sum_{I\in\mathcal{S}_{a_1}^J} \frac{\beta_{g_I}}{f(g_I)} &\propto \sum_{I\in\mathcal{S}_{a_1}^J} \left(2\gamma_1^I + \gamma_2^I + \gamma_3^I\right) + \sum_{I\in\mathcal{S}_{a_1}^{J+a_1+a_3}} \gamma_2^{I+a_1+a_3} + \sum_{I\in\mathcal{S}_{a_1}^{J+a_2}} \gamma_3^{I+a_2} , \end{aligned} \tag{3.12}$$

where $f(g) \equiv \frac{1}{16\pi^2}\frac{2g^3 C_1}{1-2C_1 g^2/16\pi^2}$. Thus, we have

$$\sum_{I\in\mathcal{S}^J_{a_1}} \frac{\beta_{g_I}}{f(g_I)} \propto \sum_{I\in\mathcal{S}^J_{a_1}} \frac{\beta_{h_I}}{h_I} + \sum_{I\in\mathcal{S}^J_{a_1}} \frac{\beta_{h'_I}}{h'_I}\,,$$

and our system of linear equations is dependent. The number of such dependencies is obviously α_1, since there are k/α_1 elements in $\mathcal{S}^J_{a_1}$. We can do the same procedure for a_2 and a_3, finding

$$\sum_{I\in\mathcal{S}^J_{a_2}} \frac{\beta_{g_I}}{f(g_I)} \propto \sum_{I\in\mathcal{S}^J_{a_2}} \frac{\beta_{h_I}}{h_I} + \sum_{I\in\mathcal{S}^J_{a_2}} \frac{\beta_{h'_I}}{h'_I}\,, \qquad \sum_{I\in\mathcal{S}^J_{a_3}} \frac{\beta_{g_I}}{f(g_I)} \propto \sum_{I\in\mathcal{S}^J_{a_3}} \frac{\beta_{h_I}}{h_I} + \sum_{I\in\mathcal{S}^J_{a_3}} \frac{\beta_{h'_I}}{h'_I}\,.$$

These three relations are not completely independent. By summing over J each of the three relations we get the same constraint. Thus, we find from here $(\sum_{i=1}^3 \alpha_i - 2)$ linear relations between the beta functions.

The β_{h_I} and $\beta_{h'_I}$ are also not completely independent: we have $\sum_I \beta_{h_I}/h_I = \sum_I \beta_{h'_I}/h'_I$, which gives another relation. Thus, all in all we have $(\sum_{i=1}^3 \alpha_i - 1)$ linear relations between the β-functions.

We have k gauge couplings, k h_I's, and k h'_I's, for a total of $3k$ parameters. We have $3k - (\sum_{i=1}^3 \alpha_i - 1)$ independent equations, so we expect to find an $(\sum_{i=1}^3 \alpha_i - 1)$ dimensional manifold of fixed points. In the generic case of $\alpha_i = 1$ we find two *exactly* marginal deformations. This is what we expect from the orbifold relations, at least for large N, as we will discuss in section 4.

We also found that some linear combinations of anomalous dimensions do not appear in (3.11), so the anomalous dimensions do not have to vanish on the fixed surface, unlike the $\mathcal{N} = 4$ case. Let us find explicitly the $(\sum_{i=1}^3 \alpha_i - 1)$-dimensional space of possible values for the anomalous dimensions. From the vanishing of (3.11) we get:

$$\begin{aligned} -\gamma_3^I &= \gamma_2^{I+a_1+a_3} + \gamma_1^{I+a_3}\,, \\ -\gamma_3^I &= \gamma_1^{I+a_2+a_3} + \gamma_2^{I+a_3}\,, \\ -\gamma_3^I - \gamma_3^{I+a_1+a_2} &= \gamma_2^{I+a_1+a_3} + \gamma_1^{I+a_2+a_3} + \gamma_1^I + \gamma_2^I\,, \end{aligned} \tag{3.13}$$

leading to

$$\begin{aligned} \gamma_2^{I+a_1+a_3} + \gamma_1^{I+a_3} &= \gamma_2^{I+a_3} + \gamma_1^{I+a_2+a_3}\,, \\ \gamma_2^{I+a_1+a_3} + \gamma_1^I &= \gamma_2^{I+a_3} + \gamma_1^{I+a_2}\,, \end{aligned} \tag{3.14}$$

and finally to

$$\gamma_1^{I+a_3} - \gamma_1^{I+a_2+a_3} = \gamma_1^I - \gamma_1^{I+a_2}\,. \tag{3.15}$$

We see that if we define $\gamma_1^I - \gamma_1^{I+a_2} \equiv K_I$ then $K_I = K_{I+a_3}$, so we have α_3 independent K_I's, which satisfy $\sum_{I\in\mathcal{S}^J_{a_2}} K_I = 0$.

Since a_2, a_3 and k have no common divisor, we find that if $\sum_{I\in\mathcal{S}^J_{a_2}} K_I = 0$ for some J then it is true for any J. Similarly, we find that $\sum_{I\in\mathcal{S}^J_{a_1}} K_I = 0$. Thus, we have one constraint on α_3 K_I's. Since

$$\begin{aligned}
\gamma_1^I - K_I &= \gamma_1^{I+a_2}\,, \\
\gamma_2^I - K_I &= \gamma_2^{I+a_1}\,, \\
-\gamma_3^I &= \gamma_1^{I+a_2+a_3} + \gamma_2^{I+a_3}\,,
\end{aligned} \tag{3.16}$$

we conclude that we have α_2 independent γ_1^I's, α_1 independent γ_2^I's and α_3-1 independent K_I's. Thus, we can have $(\sum_{i=1}^3 \alpha_i - 1)$ independent γ-functions as expected. The naive expectation is that these γ functions will not vanish at generic points on the surface of *exactly* marginal deformations.

3.1.1 Perturbative calculations

Let us now look at the perturbative conditions for vanishing β-functions. At one loop we have[8]

$$\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}\left(|h^I_{123}|^2 + |h^I_{132}|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_1}^2\right) = \gamma_1^I\,, \\
\gamma_{Q_2^I} &= \mathcal{A}\left(|h^I_{231}|^2 + |h^I_{213}|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_2}^2\right) = \gamma_2^I\,, \\
\gamma_{Q_3^I} &= \mathcal{A}\left(|h^I_{312}|^2 + |h^I_{321}|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_3}^2\right) = \gamma_3^I\,,
\end{aligned} \tag{3.17}$$

or

$$\begin{aligned}
\gamma_{Q_1^I} &= \mathcal{A}\left(|h_I|^2 + |h'_I|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_1}^2\right) = \gamma_1^I, \\
\gamma_{Q_2^I} &= \mathcal{A}\left(|h_{I-a_1}|^2 + |h'_{I+a_2}|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_2}^2\right) = \gamma_2^I, \\
\gamma_{Q_3^I} &= \mathcal{A}\left(|h_{I+a_3}|^2 + |h'_{I-a_1}|^2\right) - \mathcal{B}\left(g_I^2 + g_{I+a_3}^2\right) = \gamma_3^I.
\end{aligned} \tag{3.18}$$

Defining

$$\begin{aligned}
A_{I+a_1} &\equiv \mathcal{A}|h_I|^2 - \mathcal{B}g_{I+a_1}^2\,, \qquad B_I \equiv \mathcal{A}|h'_I|^2 - \mathcal{B}g_I^2, \\
C_I &\equiv \mathcal{B}\left(g_{I-a_1}^2 + g_{I-a_2}^2 - g_I^2 - g_{I+a_3}^2\right),
\end{aligned} \tag{3.19}$$

we have using (3.16):

$$\begin{aligned}
\gamma_{Q_1^I} &= A_{I+a_1} + B_I = \gamma_1^I\,, \\
\gamma_{Q_2^I} &= A_I + B_{I+a_2} = \gamma_2^I\,, \\
\gamma_{Q_3^{I+a_1+a_2}} &= A_{I+a_1} + B_{I+a_2} + C_{I+a_1+a_2} = -\left(\gamma_1^I + \gamma_2^I - K_I\right).
\end{aligned} \tag{3.20}$$

By subtracting the first equation from the third and summing over $\mathcal{S}^J_{a_2}$ we get zero since $\sum_{I\in\mathcal{S}^J_{a_2}} C_I = 0$, and thus we find $\sum_{I\in\mathcal{S}^J_{a_2}}(2\gamma_1^I + \gamma_2^I - K_I) = \sum_{I\in\mathcal{S}^J_{a_2}}(2\gamma_1^I + \gamma_2^I) = 0$, and further by using (3.16) we find

$$-\frac{2k}{\alpha_2}\gamma_1^J = \sum_{I\in\mathcal{S}^J_{a_2}} \gamma_2^I - 2\sum_{j=0}^{k/\alpha_2-1}\left(\frac{k}{\alpha_2} - 1 - j\right) K_{I+ja_2}\,. \tag{3.21}$$

[8]We define $\mathcal{B} = \frac{1}{16\pi^2}\frac{N^2-1}{N}$, $\mathcal{A} = \frac{1}{16\pi^2}\frac{N}{2}$.

By subtracting the second equation from the third and summing over $\mathcal{S}^J_{a_1}$ we again find zero, so $\sum_{I\in\mathcal{S}^J_{a_1}}(\gamma_1^I+2\gamma_2^I-K_I)=\sum_{I\in\mathcal{S}^J_{a_1}}(\gamma_1^I+2\gamma_2^I)=0$, and further by using (3.16) we find

$$-\frac{2k}{\alpha_1}\gamma_2^J=\sum_{I\in\mathcal{S}^J_{a_1}}\gamma_1^I-2\sum_{j=0}^{k/\alpha_1-1}\left(\frac{k}{\alpha_1}-1-j\right)K_{I+ja_1}\,. \tag{3.22}$$

Plugging (3.22) into (3.21) we find:[9]

$$\begin{aligned}\frac{4k^2}{\alpha_1\alpha_2}\gamma_1^J&=\sum_{I\in\mathcal{T}^J_{a_1,a_2}}\gamma_1^I-2\sum_{l,j}\left(\frac{k}{\alpha_1}-1-j\right)K_{J+ja_1+la_2}+\frac{4k}{\alpha_1}\sum_{j=0}^{k/\alpha_2-1}\left(\frac{k}{\alpha_2}-1-j\right)K_{J+ja_2}\\&=\sum_{I\in\mathcal{T}^J_{a_1,a_2}}\gamma_1^I-\frac{4k}{\alpha_1}\sum_{j=0}^{k/\alpha_2-1}jK_{J+ja_2}\,,\end{aligned} \tag{3.23}$$

where in the last line we used $\sum_{I\in\mathcal{T}^J_{a_1,a_2}}K_I=0$. From the first two equations in (3.20) it is clear that

$$\sum_{I\in\mathcal{S}^{J+a_2}_{a_1+a_2}}\gamma_1^{I+a_2}=\sum_{I\in\mathcal{S}^J_{a_1+a_2}}\gamma_2^I\,, \tag{3.24}$$

and that $\sum_I\gamma_1^I=\sum_I\gamma_2^I$, leading to $\sum_I\gamma_1^I=\sum_I\gamma_2^I=0$. From here and from (3.23) we see that $\gamma_1^I=-\frac{\alpha_2}{k}\sum_{j=0}^{k/\alpha_2-1}jK_{I+ja_2}$ and $\gamma_2^I=-\frac{\alpha_1}{k}\sum_{j=0}^{k/\alpha_1-1}jK_{I+ja_1}$. The periodicity of K_I now implies that $\gamma_1^I=\gamma_1^{I+a_1+a_2}$ and $\gamma_2^I=\gamma_2^{I+a_1+a_2}$, and from (3.24) we get $\gamma_1^{I+a_2}=\gamma_2^I$. Now, from (3.16) we get:

$$\gamma_1^{I+a_1+a_2}=\gamma_2^{I+a_1}=\gamma_2^I-K_I=\gamma_1^{I+a_2}-K_I=\gamma_1^I-2K_I\,, \tag{3.25}$$

so all K_I have to vanish. Using our equations this implies that all the $\gamma^I_{1,2,3}$ have to vanish. So, at one loop order we cannot turn on any non vanishing anomalous dimensions.

From the first and second equations in (3.20) we now see that $A_I=A_{I+a_1+a_2}$, so we can parameterize our solution by α_3 A_I's. The B_I's are obtained from the A_I's using (3.20). Using the second and third equations we have $A_{I+a_1}-A_I=-C_{I+a_1+a_2}$, giving the C_I's. Note that from the definition of C_I we have $\sum_{I\in\mathcal{S}^J_{a_2}}C_I,\sum_{I\in\mathcal{S}^J_{a_1}}C_I=0$, thus there are only $k+1-\alpha_1-\alpha_2$ independent C_I's (the shift by one is because $\sum_I C_I=0$ follows from both constraints). From the C_I's we get linear equations on k gauge couplings squared, whose space of solutions is $\alpha_1+\alpha_2-1$-dimensional. The h_I's and (h'_I)'s can be obtained from the B_I's, the A_I's and the gauge couplings.

Thus, to summarize, we have $\sum_i\alpha_i-1$ parameters for our solution, as expected. The solution obtained looks non generic, since in the general analysis we expected to find a $(\sum_i\alpha_i-1)$-dimensional manifold of fixed points due to the possibility of turning on non-zero anomalous dimensions. However, we see that at one loop the anomalous dimensions are forced to vanish, and nevertheless we find the expected dimensionality of the manifold of fixed points. The one-loop solution can be extended to all orders in perturbation theory, as we will illustrate below in the $\mathcal{N}=2$ example. Generally, the anomalous dimensions may be turned on in this solution at higher loop orders.

[9] We define $\mathcal{T}^J_{a_i,a_j}\equiv\{I|I=J+n_1a_i+n_2a_j;n_1,n_2\in\mathbb{N}\}$.

3.1.2 $\mathcal{N}=2$ example

We will illustrate the results of the previous section by the $\mathcal{N}=2$ example, which is the $\mathbb{C}^3/\mathbb{Z}_k$ $(1,-1,0)$ orbifold. We denote $Q_I \equiv Q_1^I$, $\tilde{Q}_I \equiv Q_2^{I+1}$ and $\Phi_I \equiv Q_3^I$.

The possible superpotentials here are:

$$\begin{aligned} W_1 &= \frac{1}{6}\mathrm{Tr}(\alpha_I \tilde{Q}_I \Phi_I Q_I + \delta_I Q_I \Phi_{I+1} \tilde{Q}_I)\,, \\ W_2 &= \frac{1}{6} h_I \mathrm{Tr}(\Phi_I \Phi_I \Phi_I)\,. \end{aligned} \tag{3.26}$$

The superpotential W_2 is specific to the $\mathcal{N}=2$ case where $a_3=0$ so we will deal with it at the end of this subsection. Setting h_I to zero, the β-functions are:

$$\begin{aligned} \beta_{g_I} &= -\frac{2g_I^3}{16\pi^2}\frac{N}{1-2Ng_I^2/16\pi^2}\left(\frac{1}{2}\left(\gamma_{Q_I}+\gamma_{\tilde{Q}_I}+\gamma_{Q_{I-1}}+\gamma_{\tilde{Q}_{I-1}}\right)+\gamma_{\Phi_I}\right), \\ \beta_{\alpha_I} &= \alpha_I(\gamma_{Q_I}+\gamma_{\tilde{Q}_I}+\gamma_{\Phi_I})\,, \\ \beta_{\delta_I} &= \delta_I(\gamma_{Q_I}+\gamma_{\tilde{Q}_I}+\gamma_{\Phi_{I+1}})\,. \end{aligned} \tag{3.27}$$

Because of the symmetry of all interactions we have $\gamma_{Q_I} = \gamma_{\tilde{Q}_I}$. Equating the β-functions to zero we obtain that $\forall I : \gamma_{\Phi_I} \equiv \gamma$, and also all the γ_{Q_I}'s have to be equal and equal to $-\frac{1}{2}\gamma$. So, we have here $3k$ couplings and one possible independent anomalous dimension γ. Since a priori we have $2k$ different anomalous dimensions, we have $2k-1$ equations, and we expect to find a $(k+1)$-dimensional manifold of fixed points.

Next, we do the perturbative analysis. The one-loop calculation gives:

$$\begin{aligned} \gamma_{\Phi_I} &= \frac{1}{16\pi^2}\frac{N}{4}\left(\left(|\delta_{I-1}|^2+|\alpha_I|^2\right)-8g_I^2\right), \\ \gamma_{Q_I} &= \frac{1}{16\pi^2}\frac{N^2-1}{4N}\left\{\left(|\delta_I|^2+|\alpha_I|^2\right)-4\left(g_I^2+g_{I+1}^2\right)\right\}. \end{aligned} \tag{3.28}$$

Defining $B_I \equiv |\delta_{I-1}|^2 - 4g_I^2$, $A_I \equiv |\alpha_I|^2 - 4g_I^2$ and $16\pi^2\frac{4N}{N^2-1}\gamma \to \gamma$, the requirement of vanishing β-functions becomes :

$$\begin{aligned} B_I + A_I &= \frac{N^2-1}{N^2}\gamma\,, \\ B_{I+1} + A_I &= -\frac{1}{2}\gamma\,. \end{aligned} \tag{3.29}$$

By subtracting the first line from the second and summing over I, we find that $\gamma=0$. Thus, again we find that at one loop precision the γ parameter has to vanish. As we will see later this is not necessarily true for higher loop calculations.

The case of vanishing γ is the case of vanishing anomalous dimensions. We see that in this case the condition for having zero β-functions is that for all I, $B_I = X = -A_I$ for some number X which is a parameter. Thus, we find a family of solutions parameterized by X and the gauge couplings, with

$$\begin{aligned} |\delta_{I-1}|^2 &= X + 4g_I^2\,, \\ |\alpha_I|^2 &= 4g_I^2 - X\,. \end{aligned} \tag{3.30}$$

We see that the parameter X is constrained to the range $-\min_I\left\{4g_I^2\right\} \leq X \leq \min_I\left\{4g_I^2\right\}$. The case $X = 0$ is the case of $\mathcal{N} = 2$ SUSY.

To summarize, we find a $(k+1)$-dimensional space of solutions. We expected a $(k+1)$-dimensional manifold from the general analysis, but the $(+1)$ was due to the γ parameter. At one loop we find that $\gamma = 0$ but nevertheless we have a $(k+1)$-dimensional space of solutions.

A natural question is whether the vanishing of γ extends to higher loops, and whether we can extend our solution, parameterized by the gauge couplings and the parameter X, to higher loops. We will prove that the non-vanishing X solution does not disappear at higher loops. First we will represent a general solution as a function of the gauge couplings and the X parameter. The procedure we use here is similar to the coupling constant reduction procedure described in [18].

The most general solution for α_I and δ_I depending on our parameters X and g_I, consistent with the one loop analysis and with the $\mathcal{N} = 2$ case (which is known to be exactly conformal for any gauge couplings), is of the form

$$\begin{aligned} |\delta_{I-1}|^2 &= 4g_I^2 + X\left(1 + \sum_{m,j,l_s} a^{(I)m}_{l_1,\ldots,l_j} X^m g_{l_1}^2, \ldots, g_{l_j}^2\right), \\ |\alpha_I|^2 &= 4g_I^2 - X\left(1 + \sum_{m,j,l_s} b^{(I)m}_{l_1,\ldots,l_j} X^m g_{l_1}^2, \ldots, g_{l_j}^2\right), \end{aligned} \tag{3.31}$$

where a, b are some constants and $m+j > 0$. We will construct the solution by an inductive process. Assume that we have computed the a's and b's in these solutions up to $(n-1)$'th order in g^2 and X, and look at the n'th order. First, we calculate the γ_{Q_I} and γ_{Φ_I}. We can write them as:

$$\begin{aligned} \gamma_{Q_I}^{(n)} &= \gamma_{Q_I}^{(n)(\text{1-loop})} + \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} \\ \gamma_{\Phi_I}^{(n)} &= \gamma_{\Phi_I}^{(n)(\text{1-loop})} + \gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}. \end{aligned} \tag{3.32}$$

We define:

$$\begin{aligned} \tilde{B}_I^{(n)} &\equiv (\delta_{I-1}^2)^{(n)} = X \cdot \left(\sum_{m,j,l_s,m+j=(n-1)} a^{(I)m}_{l_1,\ldots,l_j} X^m g_{l_1}^2, \ldots, g_{l_j}^2\right), \\ \tilde{A}_I^{(n)} &\equiv (\alpha_I^2)^{(n)} = -X \cdot \left(\sum_{m,j,l_s,m+j=(n-1)} b^{(I)m}_{l_1,\ldots,l_j} X^m g_{l_1}^2, \ldots, g_{l_j}^2\right). \end{aligned} \tag{3.33}$$

The $\gamma_{Q_I}^{(\text{1-loop})}$ and $\gamma_{\Phi_I}^{(\text{1-loop})}$ have a special structure, giving:

$$\begin{aligned} \tilde{B}_{I+1}^{(n)} + \tilde{A}_I^{(n)} &= \gamma_{Q_I}^{(n)(\text{1-loop})}, \\ \tilde{B}_I^{(n)} + \tilde{A}_I^{(n)} &= \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(\text{1-loop})}. \end{aligned} \tag{3.34}$$

Now, we parameterize the remaining contributions to the γ's as:

$$\begin{aligned}
\gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} &\equiv T_I^{(n)} + S_{I+1}^{(n)} - \frac{1}{2}\tilde{\gamma}^{(n)}\,,\\
\frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})} &\equiv T_I^{(n)} + S_I^{(n)} + \frac{N^2-1}{N^2}\tilde{\gamma}^{(n)}\,,
\end{aligned} \tag{3.35}$$

where the different quantities are defined as :

$$\begin{aligned}
-k\left(\frac{1}{2}+\frac{N^2-1}{N^2}\right)\tilde{\gamma}^{(n)} &\equiv \sum_I \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} - \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}\,,\\
\Delta X^{(n)} &= S_1^{(n)} \equiv 0\,,\\
\gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} - \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})} &= S_{I+1}^{(n)} - S_I^{(n)} - \left(\frac{1}{2}+\frac{N^2-1}{N^2}\right)\tilde{\gamma}^{(n)}\,.
\end{aligned} \tag{3.36}$$

Note that $\Delta X^{(n)}$ is just a redefinition of X, so we can set it to zero without any loss of generality, and the $T_I^{(n)}$'s are automatically determined from above. We see that the definitions above uniquely determine S_I, T_I and $\tilde{\gamma}$.

The crucial point is that in order to calculate the one loop contribution to the n'th order we use the n'th order components of (3.31), while for two loops we use the (n-1)'th order of (3.31), and so on. Thus, because we have already determined (3.31) up to $n-1$'th order, $\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}$ and $\gamma_{Q_I}^{(n)(2\ldots n\text{-loops})}$ depend only on already determined quantities. The yet undetermined quantities appear only at one loop.

From the β-function analysis we know that $\gamma_{Q_I} = -\frac{1}{2}\gamma$ and $\gamma_{\Phi_I} = \gamma$. Thus, using the equations above we find

$$\begin{aligned}
\tilde{A}_I^{(n)} &= -T_I^{(n)}\,,\\
\tilde{B}_I^{(n)} &= -S_I^{(n)}\,,\\
\tilde{\gamma}^{(n)} &= \gamma^{(n)}\,,
\end{aligned} \tag{3.37}$$

where in the first two lines we are determining the n'th order a's and b's, and in the third line we are computing the γ parameter. We see that it does not have to be zero at higher loops. This procedure is well defined and unique, and can be extended to any order in perturbation theory. So, we have proven that there exists a $(k+1)$-dimensional manifold of fixed points parameterized by the gauge couplings and the X parameter,[10] at all orders of perturbation theory.

The procedure described above for the $\mathcal{N}=2$ theory can be repeated for any general orbifold theory. In the general case we also find that the anomalous dimensions can not be turned on at one loop order, but nevertheless the number of *exactly* marginal directions is as predicted from the general analysis. The anomalous dimensions may be turned on at higher orders of perturbation theory.

[10] Equivalently, we can use γ to parameterize the solution, if it is non-zero starting from some order in the perturbation series.

As mentioned above, in the specific $\mathcal{N}=2$ example we can also turn on a deformation $W=\frac{1}{6}h_I\mathrm{Tr}(\Phi_I\Phi_I\Phi_I)$, which we now analyze. In this case we are constrained to have $\gamma_{\Phi_I}=0$ or equivalently $\gamma=0$. We have here $4k$ couplings (k gauge couplings, α_I's, δ_I's and h_I's), and $2k$ constraints $\gamma_{\Phi_I}=\gamma_{Q_I}=0$, so we expect naively to find a $2k$-dimensional manifold of fixed points.

At one-loop we get, defining $C_I\equiv\frac{1}{8}\frac{N^2-4}{N^2}|h_I|^2$,

$$\begin{aligned} B_I+A_I &= -C_I\,, \\ B_{I+1}+A_I &= 0\,. \end{aligned} \tag{3.38}$$

By subtracting the equations and summing over I we get $\sum_I C_I=0$, but this is impossible unless all h_I vanish because C_I is positive definite. So, we conclude that there are no fixed points with non vanishing h_I at one-loop.

We now proceed in search of all loop solutions like we did above. The general expressions for α_I and δ_I can now depend also on the h_I's, and we proceed as before:

$$\begin{aligned} \gamma_{Q_I}^{(n)} &= \gamma_{Q_I}^{(n)(\text{1-loop})}+\gamma_{Q_I}^{(n)(2\ldots n\text{-loops})}\,, \\ \gamma_{\Phi_I}^{(n)} &= \gamma_{\Phi_I}^{(n)(\text{1-loop})}+\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}\,, \end{aligned} \tag{3.39}$$

where the $\gamma_{Q_I}^{(\text{1-loop})}$ and $\gamma_{\Phi_I}^{(\text{1-loop})}$ have a special structure,

$$\begin{aligned} \tilde{B}_{I+1}^{(n)}+\tilde{A}_I^{(n)} &= \gamma_{Q_I}^{(n)(\text{1-loop})}\,, \\ \tilde{B}_I^{(n)}+\tilde{A}_I^{(n)}+C_I^{(n)} &= \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(\text{1-loop})}\,. \end{aligned} \tag{3.40}$$

Now, we parameterize the remaining contributions to the γ's as

$$\begin{aligned} \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} &\equiv T_I^{(n)}+S_{I+1}^{(n)}\,, \\ \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})} &\equiv T_I^{(n)}+S_I^{(n)}-C_I^{(n)}\,, \end{aligned} \tag{3.41}$$

where the different quantities are defined by

$$\begin{aligned} \sum_I C_I^{(n)} &= \sum_I \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})}-\frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}\,, \\ \Delta X^{(n)} &= S_1^{(n)}(\equiv 0)\,, \\ \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})}-\frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})} &= S_{I+1}^{(n)}-S_I^{(n)}+C_I^{(n)}\,. \end{aligned} \tag{3.42}$$

Again, $\Delta X^{(n)}$ is just a redefinition of X, so we can set it to zero without any loss of generality, the first equation determines $\sum_I C_I$ and we can choose the individual C_I's arbitrarily (subject to this constraint), and then the $T_I^{(n)}$'s are automatically determined from above.

The definitions above are well-defined, except for one caveat. The first equation above cannot always be satisfied: in the lowest order where $C_I^{(n)}$ is not zero it has to be positive, so at that order $\hat{\gamma} \equiv \sum_I \gamma_{Q_I}^{(n)(2\ldots n\text{-loops})} - \frac{N^2-1}{N^2}\gamma_{\Phi_I}^{(n)(2\ldots n\text{-loops})}$ has to be positive. In [15] we computed this combination $\hat{\gamma}$ up to three loop order (without the h_I's) and found that it vanished, thus implying that no h_I's can be turned on up to this order.

If at some higher order we find that $\hat{\gamma}$ is negative then there are no perturbative solutions with non-zero h_I. On the other hand, if at the first order where it is non-zero it is positive, then we proceed like in the previous case to obtain a solution, by demanding that $\gamma_{Q_I} = \gamma_{\Phi_I} = 0$, so that

$$\begin{aligned} \tilde{A}_I^{(n)} &= -T_I^{(n)}\,, \\ \tilde{B}_I^{(n)} &= -S_I^{(n)}\,. \end{aligned} \tag{3.43}$$

This defines the yet undetermined a's and b's, and the equation for C_I gives one constraint on the k h_I couplings, giving an additional $(k-1)$-dimensional space of solutions as expected from the general analysis. Again, this procedure is well defined and can be extended to any order in perturbation theory, if the sign of $\hat{\gamma}$ is right.

3.2 Special cases

The analysis of other special cases is analogous to our analysis above, and we will present here only the results (details may be found in [15]).

- $k = 3$, $\vec{a} = (1, -1, 0)$

In addition to both types of deformations discussed in the previous subsection we can have here also a superpotential $W = \frac{\kappa}{3!}\mathrm{Tr}(Q_1Q_2Q_3) + \frac{\tilde{\kappa}}{3!}\mathrm{Tr}(\tilde{Q}_1\tilde{Q}_2\tilde{Q}_3)$. In this case we find, both in the general analysis and in the one-loop analysis, 7 *exactly* marginal deformations, which is three deformations more than the general $\mathcal{N} = 2$ case we treated above.

- $k = 3$, $\vec{a} = (1, 1, 1)$

In this case we can also have a superpotential of the form $W = h^I_{ijk}\mathrm{Tr}(Q_i^I Q_j^{I+1} Q_k^{I+2})$, and we find 3 *exactly* marginal deformations, one beyond the number of deformations expected from the generic case.

- $\vec{a} = (a, a, -2a)$

This case includes theories with $\mathcal{N} = 2$ SUSY ($k = 2$, $\vec{a} = (1, -1, 0)$) and theories with $\mathcal{N} = 1$ SUSY. The extra interactions that can be turned on here are h^I_{113}, h^I_{223} from (3.5). We define $p_I \equiv h^I_{113}$ and $s_I \equiv h^I_{223}$. The extra β-functions are:

$$\begin{aligned} \beta_{s_I} &\propto \gamma_2^I + \gamma_2^{I+a} + \gamma_3^{I+2a}\,, \\ \beta_{p_I} &\propto \gamma_1^I + \gamma_1^{I+a} + \gamma_3^{I+2a}\,. \end{aligned} \tag{3.44}$$

From here one can obtain that in the odd k case all the anomalous dimensions γ_1^I, γ_2^I are equal, and in the even k case we have two independent anomalous dimensions : $\gamma_{1,2}^{I+2a} = \gamma_{1,2}^I$. We also see that all the γ_3^I's are equal and equal to $-(\gamma_1^I + \gamma_1^{I+a})$.

In the even k case we find a new solution, not obtained in the general case. This solution is parameterized by three couplings, which can be chosen to be the gauge coupling (equal for all groups) and two of the p_I's ($p_I = p_{I+2a}$).

- $k = 3k'$, $\vec{a} = (a, k/3 - a, -k/)$ and $k = 6$, $\vec{a} = (1, 1, 4)$.

In these cases we can turn on superpotentials of the form $h_{333}^I \mathrm{Tr}(Q_3^I Q_3^{I+2k/3} Q_3^{I+4k/3})$. We add k' couplings but we also add k' new β-functions,

$$\beta_{h_{333}^I} \propto \gamma_3^I + \gamma_3^{I+k'} + \gamma_3^{I+2k'} = 0\,, \tag{3.45}$$

or $\sum_{I\in\mathcal{S}_{2k/3}^J} \gamma_3^I = 0$. Thus we do not expect any new *exactly* marginal directions here and we indeed don't see them in the perturbative calculation.

- $\mathrm{SU}(N = 3)$

As we mentioned, we can have here a larger class of marginal deformations (3.7). Again we discuss the example of the $\mathcal{N} = 2$ theory and briefly report the results for other cases. In this case one can add additional superpotentials of the form $\frac{\rho_I}{3!}\epsilon_{lmn}\epsilon^{abc}(Q_I)_a^l(Q_I)_b^m(Q_I)_c^n$ and $\frac{\tilde{\rho}_I}{3!}\epsilon_{lmn}\epsilon^{abc}(\tilde{Q}_I)_a^l(\tilde{Q}_I)_b^m(\tilde{Q}_I)_c^n$, and

$$\begin{aligned} \beta_{\rho_I} &= 3\rho_I \cdot \gamma_{Q_I}\,, \\ \beta_{\tilde{\rho}_I} &= 3\tilde{\rho}_I \cdot \gamma_{\tilde{Q}_I}\,. \end{aligned} \tag{3.46}$$

From here and (3.27) we find for all I $\gamma_{Q_I} = \gamma_{\tilde{Q}_I} = \gamma_{\Phi_I} = 0$. In this case adding non zero h_I's doesn't change the conditions for the γ's (γ_{Φ_I} has to be zero anyway), so we can consider them together. We have here $6k$ couplings and $3k$ conditions, leading naively to a $3k$ dimensional manifold of fixed points. The interactions we add affect the one loop γ_{Φ_I} as in (3.38), with additional terms $2\rho_I^2(\equiv K_I)$ for Q and $2\tilde{\rho}_I^2(\equiv \tilde{K}_I)$ for $\tilde{Q}$:

$$\begin{aligned} B_I + A_I &= -C_I\,, \\ B_{I+1} + A_I &= -K_I\,, \\ B_{I+1} + A_I &= -\tilde{K}_I\,. \end{aligned} \tag{3.47}$$

At one loop we find that necessarily

$$\begin{aligned} \sum_I C_I &= \sum_I K_I\,, \\ K_I &= \tilde{K}_I\,, \end{aligned} \tag{3.48}$$

and the one loop solution is:

$$\begin{aligned} |\delta_{I-1}|^2 &= X + 4g_I^2 + \sum_{J=1}^{I-1}(C_J - K_J)\,, \\ |\alpha_I|^2 &= 4g_I^2 - X + \sum_{J=1}^{I-1} K_J - \sum_{J=1}^{I} C_J\,. \end{aligned} \tag{3.49}$$

The general solution is parameterized by k gauge couplings, the X parameter, k ρ_I's and k h_I's subject to the condition (3.48) above, giving a total of $3k$ parameters, and a $3k$-dimensional manifold of fixed points as expected. These solutions can be extended to all loops like we did before.

The number of *exactly* marginal deformations in other cases with $\mathrm{SU}(N=3)$ gauge group is :

- General case: $3k$ *exactly* marginal deformations.
- $k=3$, $\vec{a}=(1,-1,0)$: 11 *exactly* marginal deformations.
- $k=3$, $\vec{a}=(1,1,1)$: 33 *exactly* marginal deformations.
- $\vec{a}=(a,a,-2a)$: $5k$ *exactly* marginal deformations.
- $\vec{a}=(a,k/3-a,-k/3)$, $k=3k'$: $4k$ *exactly* marginal deformations.
- $\vec{a}=(1,1,4)$, $k=6$: 36 *exactly* marginal deformations.

4. Summary and discussion

First we summarize our results on the *exactly* marginal deformations :

- $\mathcal{N}=4$

  We found that the only supersymmetric *exactly* marginal deformations of $\mathcal{N}=4$ SYM, other than changing the gauge coupling, are the superpotentials :

$$\begin{aligned} &\delta\lambda\epsilon_{ijk}\mathrm{Tr}\left(\Phi^i\left[\Phi^j,\Phi^k\right]\right),\\ &\sum_i h\mathrm{Tr}\left(\Phi^i\left\{\Phi^i,\Phi^i\right\}\right),\\ &h_{123}\mathrm{Tr}\left(\Phi^1\left\{\Phi^2,\Phi^3\right\}\right), \end{aligned} \tag{4.1}$$

  with one equation relating λ, h_{123}, h and the gauge coupling. These fixed points are IR stable, and the theory is not asymptotically free for any choice of the coupling constants.

- $\mathcal{N}=2$

  In the $\mathcal{N}=2$ SYM theories obtained from $\mathbb{C}^2/\mathbb{Z}_k$ orbifolds we found :

  - General k

    We found that there is one *exactly* marginal direction (in addition to the k gauge couplings), parameterized by a parameter X, which can be proven to be *exactly* marginal at any order of perturbation theory. The $X=0$ case has

$\mathcal{N} = 2$ SUSY and all the γ's vanish, however if $X \neq 0$ there may be non-zero γ's along this direction. From the general analysis we expect to have here another $k-1$ *exactly* marginal directions, related to turning on $\mathrm{Tr}(\Phi_I \Phi_I \Phi_I)$ superpotentials. We do not see these *exactly* marginal directions up to three loops. This, however, does not necessarily prevent them from appearing at higher loops. Whether they appear or not depends on the value of a linear combination of the γ_{Q_I} and the γ_{Φ_I} along the flat direction parameterized by X, which does not include these operators. If this linear combination is positive or zero, then these marginal directions are ruled out, but if it is negative then these exactly marginal directions exist. In any case, the total number of *exactly* marginal directions is at least $k+1$.

– $k = 3$

In this case we have again the gauge couplings and the X deformation, as above, and we can also have three additional *exactly* marginal directions. The perturbative result agrees with the general analysis, and we see all the marginal deformations already at one loop. The total number of *exactly* marginal directions here is 7.

– $\mathrm{SU}(N = 3)$

Here we have yet a larger space of deformations : in addition to the gauge couplings and the X deformation, we get $2k-1$ additional deformations, for a total of $3k$ *exactly* marginal deformations for general k, and 11 for $k = 3$. Again we see all the deformations already at one-loop, in agreement with the general analysis.

• $\mathcal{N} = 1$

For the theory coming from a $\mathbb{Z}_k$ orbifold with general (a_1, a_2, a_3), denoting the largest common divisor of a_i with k by α_i, we show that the number of *exactly* marginal directions is $\sum_i \alpha_i - 1$. In the case where two of the a_is are equal and k is even we get additional *exactly* marginal directions. In the special case of $\mathrm{SU}(N = 3)$ we get much larger manifolds of fixed points, ranging from dimension $3k$ in the most general (a_1, a_2, a_3) $\mathbb{Z}_k$ theory, to $11k$ in the $k = 3$ case.

In the large N limit, there is a strong relation between the orbifold theories and the parent $\mathcal{N} = 4$ theory. Operators in the $\mathcal{N} = 4$ theory which are invariant under the orbifold action are related to "untwisted operators" in the orbifold theories; for example, the operator $\mathrm{Tr}(F_{\mu\nu}^2)$ in the $\mathcal{N} = 4$ theory is related to the operator $\sum_I \mathrm{Tr}((F_{\mu\nu}^I)^2)$ in the orbifold theories. The relation is [9, 10] that in the large N limit, where only planar diagrams contribute to correlation functions, all correlation functions of these operators are equal (up to some powers of k) in the "parent" theory and in the orbifold theories.

Clearly, this implies that in the large N limit, *exactly* marginal deformations of the $\mathcal{N} = 4$ theory which are invariant under the orbifolding should correspond to exactly marginal deformations of the orbifold theories as well. Our analysis above shows that this is, in fact, true even at finite N. Two of the *exactly* marginal deformations of the $\mathcal{N}=4$ theory are always invariant under the orbifolds we perform; the only one which is not is $W = h\sum_i \mathrm{Tr}((\Phi^i)^3)$. And indeed, in all our orbifold theories we find two *exactly* marginal deformations coming from these deformations. Such *exactly* marginal deformations are necessarily invariant under the action of the $\mathbb{Z}_k$ orbifold group. For example, in the $\mathcal{N}=2$ case, these two deformations are the equal change in all the gauge couplings, and the deformation parameterized by X. In the special $k=3$ case, the third *exactly* marginal deformation of $\mathcal{N}=4$ SYM is also invariant under the orbifolding, and indeed we find this additional deformation in all the cases with $k=3$. So, our results are consistent with the expectations from the orbifold point of view.

The theories we describe here all have string theory duals, given by type-IIB string theory on $AdS_5 \times S^5/\mathbb{Z}_k$ [1, 2]. *Exactly* marginal deformations correspond to moduli of these string compactifications, in which some massless scalar fields on AdS_5 acquire VEVs. The string theory dual of the *exactly* marginal deformations of $\mathcal{N}=4$ SYM will be discussed, in the large N supergravity approximation, in [16]. When the deformation is invariant under $\mathbb{Z}_k$, one can orbifold also the deformed backgrounds, and this will give the string theory backgrounds corresponding to the *exactly* marginal deformations of the orbifold theories. Alternatively, they can be described by turning on untwisted sector moduli fields in $AdS_5 \times S^5/\mathbb{Z}_k$.

The additional *exactly* marginal deformations that we find in the orbifold theories are all related to twisted sector moduli in the string theory. Such moduli can only appear on the string theory side if the orbifold action has fixed points on the S^5, at which massless twisted sector fields are located. For example, in the $\mathcal{N}=2$ case, the orbifold keeps one direction in $\mathbb{C}^3$ fixed, and thus because the orbifold acts on the S^5 factor of the $AdS_5 \times S^5$ as it acts on the angular coordinates of the $\mathbb{R}^6 \sim \mathbb{C}^3$, we have a fixed S^1 in the S^5. This enables the appearance of massless twisted sector states which can correspond to some *exactly* marginal operators on the field theory side. In this case these $k-1$ deformations correspond to $k-1$ blow up modes of the $\mathbb{C}^2/\mathbb{Z}_k$ singularity [5] (more precisely, they correspond to the 2-form fields on the vanishing 2-cycles in these singularities). Another case of *exactly* marginal operators coming from the twisted sector is the 2 additional operators we get in the $\mathcal{N}=2$ $\mathbb{Z}_3$ case.

In the $\mathcal{N}=1$ case the only fixed point of the full $\mathbb{Z}_k$ action is the origin of $\mathbb{C}^3$. However, we still can have massless twisted sector states if some of the elements of the orbifold group have fixed points on the S^5. The action of the orbifold is given by :

$$\wp \equiv \begin{pmatrix} e^{\frac{2\pi i}{k}a_1} & 0 & 0 \\ 0 & e^{\frac{2\pi i}{k}a_2} & 0 \\ 0 & 0 & e^{\frac{2\pi i}{k}a_3} \end{pmatrix}. \tag{4.2}$$

Now, if α_1 is the largest common divisor of k and a_1, and if we start with the vector $(1,0,0)$, we will get back to our starting point after k/α_1 applications of $\wp$. So, if $\alpha_1 > 1$, the k/α_1'th twisted sector has fixed points and could include massless states. More generally we find $\alpha_i - 1$ twisted sectors which could have massless states living on a circle on S^5.

In the general theory we found $\sum_i \alpha_i - 1$ *exactly* marginal deformations, two of which came from the untwisted sector. Thus, we see that each twisted sector with fixed points contributes precisely one *exactly* marginal deformation to the theory, corresponding to turning on a massless scalar in this twisted sector. The $\sum_i \alpha_i - 3$ *exactly* marginal deformations coming from the twisted sectors can be related to the $\mathcal{N} = 2$ blow up modes – locally near the fixed line where the light twisted sector states live, the background looks like the $\mathcal{N} = 2$ $AdS_5 \times S^5/\mathbb{Z}_{\alpha_i}$ theory, and these modes will correspond to 2-form fields on vanishing 2-cycles.

So, in the general case we have a nice interpretation for all the deformations, as coming from untwisted or twisted sectors in the string theory, with one deformation in each twisted sector with fixed points. For SU($N = 3$) we get a much larger space of deformations. It would be interesting to understand the origin of this larger space of deformations on the string theory side. Of course, supergravity is not a good approximation in such a case, so this would require a full understanding of string theory on $AdS_5 \times S^5/\mathbb{Z}_k$. In some cases we also find additional *exactly* marginal deformations coming the twisted sector : the 2 extra deformations of the $\mathcal{N} = 2$ $\mathbb{Z}_3$ theory and the $(a, a, -2a)$ extra deformations, for instance. In such cases additional massless twisted sector fields, beyond the one complex scalar of the general case, should also correspond to *exactly* marginal deformations. It would be interesting to understand the form of these *exactly* marginal deformations on the string theory side.

An interesting point which we have not been able to resolve involves the marginal operators of the form $\mathrm{Tr}(\Phi_I^3)$ in the general $\mathcal{N} = 2$ $\mathbb{Z}_k$ case. From the general analysis, by counting the number of variables and equations, $k-1$ of these operators (which correspond to some twisted sector fields) are expected to be *exactly* marginal. However, we saw that these operators do not appear in perturbation theory up to 3-loop order. Higher loop calculations are required to decide if these exactly marginal deformations appear in the weak coupling region or not. It would be interesting also to analyze these marginal deformations, using string theory, in the strong coupling regime, and to see if *exactly* marginal deformations appear there or not.

Acknowledgments

The research of O.A. was supported in part by the Israel-U.S. Binational Science Foundation, by the IRF Centers of Excellence program, by the European RTN network HPRN-CT-2000-00122, and by Minerva.

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